

❏ Moments of Inertia

①

- Recall center of mass

$$\bar{x} = \frac{M_y}{M} \quad \bar{y} = \frac{M_x}{M}$$

$$M_y = \iint_R x \delta(x, y) dA$$

$R \curvearrowright$ distance to y -axis

$$M_x = \iint_R y \delta(x, y) dA$$

$R \curvearrowright$ distance to x -axis

- Moments of Inertia I_x, I_y

$$I_x = \iint_R y^2 \delta(x, y) dA$$

$$I_y = \iint_R x^2 \delta(x, y) dA$$

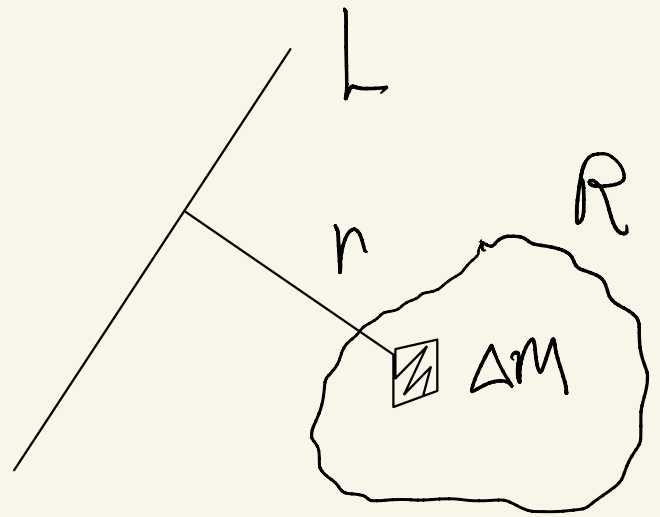
int density
by distance
to axis
squared

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Defn: Moment of Inertia
about an arbitrary axis L

$$I_L = \iint_R r^2 \delta(x, y) dA$$

Here $r = r(x, y)$
is a function of



(x, y) which must be computed
to evaluate the integral.

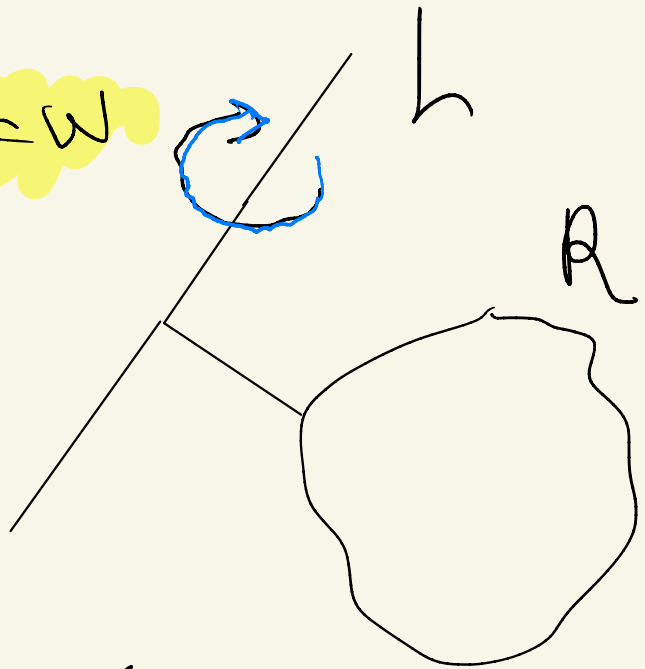
Q: Why are moments of Inertia
important?

Ans: (For us) Kinetic Energy of
Rotation

Ex: Derive a formula for the kinetic Energy stored in a metal plate R of density $\delta(x,y)$ when it is rotated about axis L

Picture:

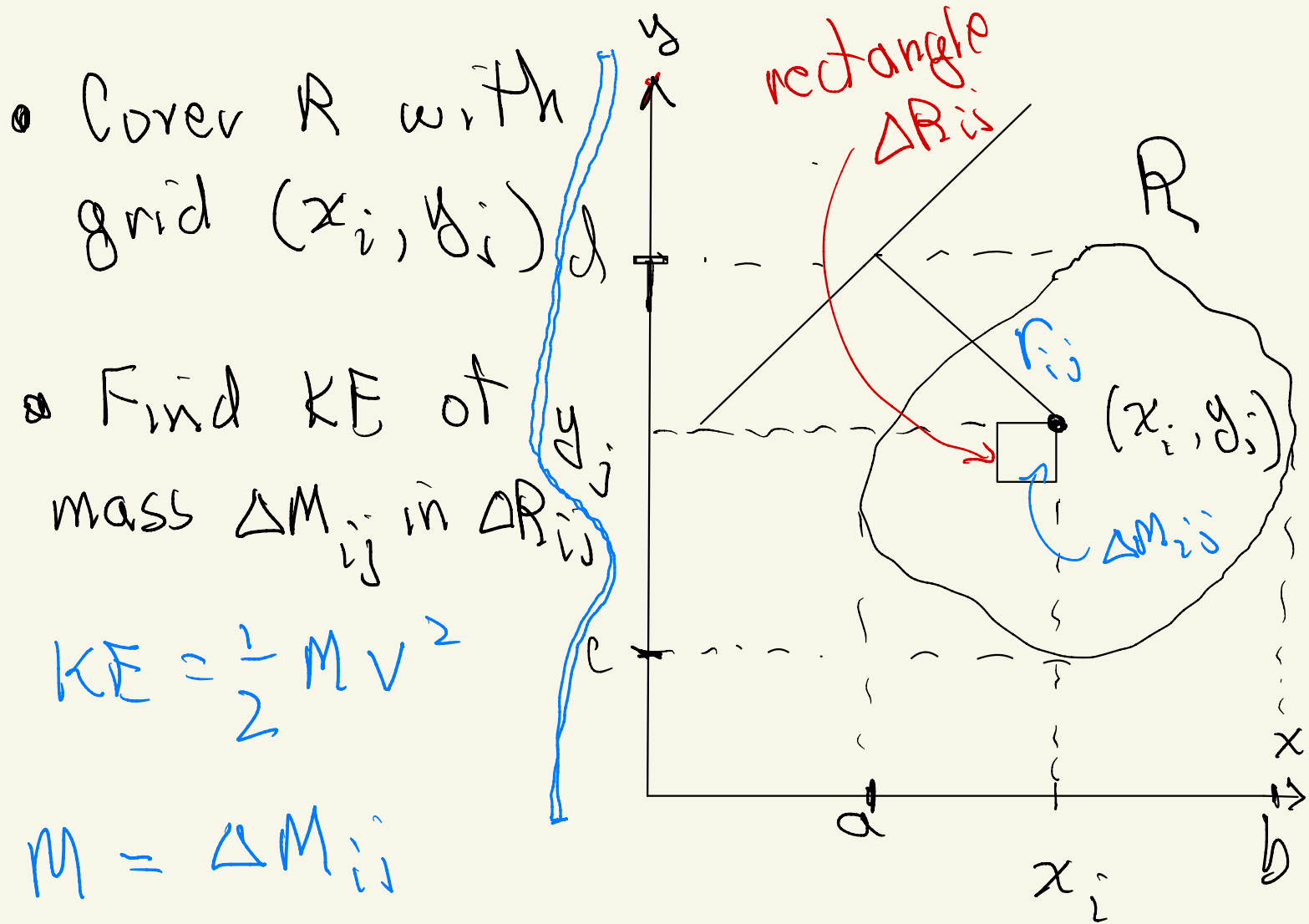
$$\frac{d\theta}{dt} = \omega$$



Soln: We start with the defn of Kinetic Energy (KE) of a point mass m :

Defn $KE = \frac{1}{2} m v^2$

Soln: Idea: approx the KE of 4
the continuously distributed $\delta(x, y)$
by a Riemann Sum & take the
limit to get an integral...



$$\Delta KE_{ij} = \frac{1}{2} \underbrace{\Delta M_{ij}}_M \underbrace{(r_{ij} \omega)^2}_{v^2}$$

$r_{ij} = r_{ij}(x,y)$ ⑤

$$\Delta M_{ij} \approx \delta(x_i, y_i) \Delta x \Delta y$$

$\frac{\text{mass}}{\text{area}} \times \text{area}$

ΔM_{ij}

$$\Delta KE_{ij} = \frac{1}{2} \delta(x_i, y_i) r_{ij}^2 \omega^2 \Delta x \Delta y$$

constant

constant angular rotation rate

$$KE = \lim_{N \rightarrow \infty} \sum_{(x_i, y_i) \in R} \left(\frac{1}{2} \omega^2 \right) \delta(x_i, y_i) r_{ij}^2 \Delta x \Delta y$$

$$KE = \frac{1}{2} \omega^2 \iint_R r^2 \delta(x, y) dx dy = \frac{1}{2} I_L \omega^2$$

Conclude: The Kinetic Energy of rotation of a continuously distributed mass (metal plate) about axis L is :

$$KE = \frac{1}{2} I_L \omega^2$$

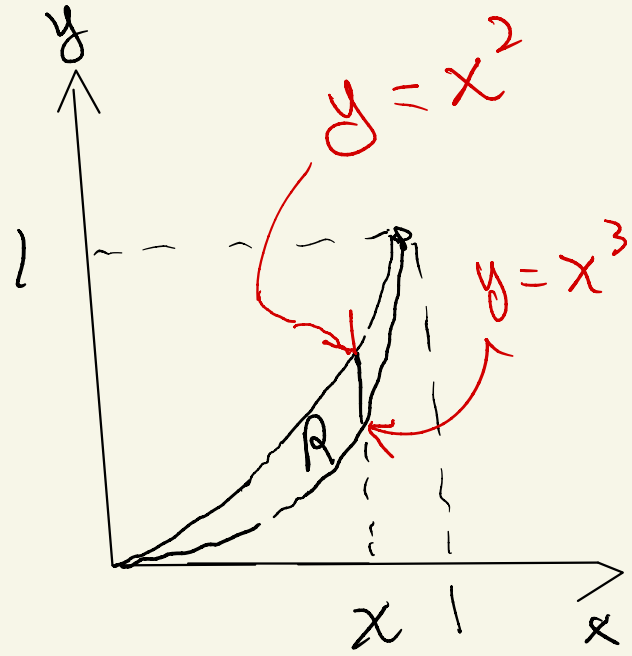
Moment of
Inertia
about
axis L

angular
rotation
rate
 $\omega = \frac{\text{radians}}{\text{sec}}$

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Example: Consider a metal plate R of density $\delta(x,y) = 1$ whose shape is the region between $y = x^2$ and $y = x^3$

- Find the center of mass —



$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

$$M = \iint_R \delta(x,y) dA = \int_0^1 \int_{x^3}^{x^2} 1 \, dy \, dx$$

$$= \int_0^1 \left[y \right]_{x^3}^{x^2} dx = \int_0^1 (x^2 - x^3) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$M = \frac{1}{12}$$

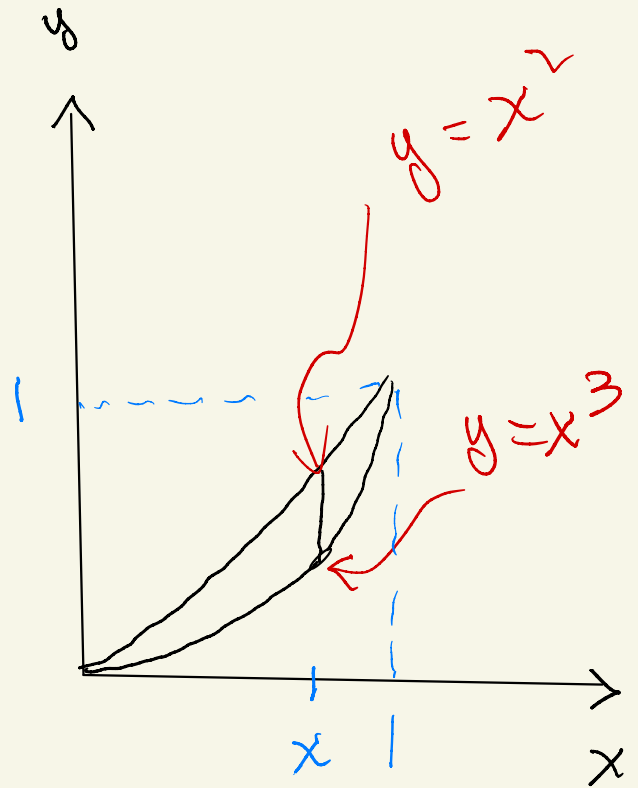
$$M_y = \iint_R x \delta(x, y) dA$$

$$= \int_0^1 \int_{x^3}^{x^2} x \cdot 1 dy dx$$

$$= \int_0^1 x y \Big|_{y=x^3}^{x^2} dx$$

$$= \int_0^1 x (x^2 - x^3) dx = \int_0^1 x^3 - x^4 dx$$

$$= \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$



$$M = \frac{1}{12} \quad M_y = \frac{1}{20}$$

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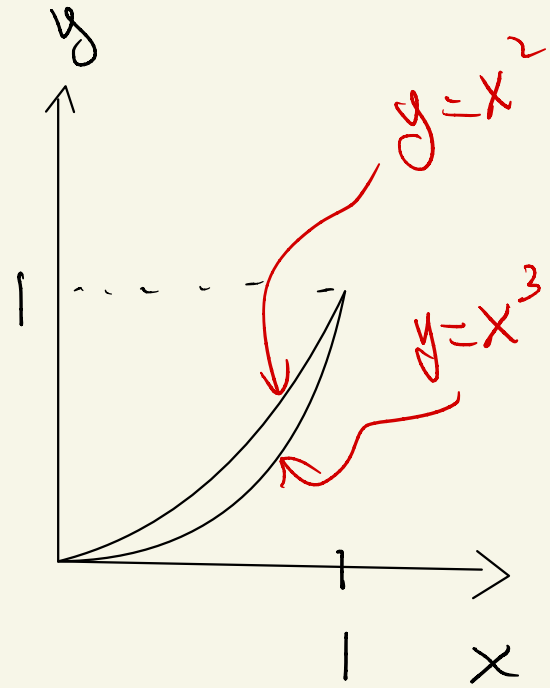
$$M_x = \iint_R y \delta(x, y) dA$$

$$= \iint_R y \cdot dA$$

$$= \int_0^1 \int_{x^3}^{x^2} y dy dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^3}^{y=x^2} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} - \frac{x^6}{2} \right) dx = \left[\frac{x^5}{10} - \frac{x^7}{14} \right]_0^1$$

$$= \frac{1}{10} - \frac{1}{14} = \frac{7}{70} - \frac{5}{70} = \frac{2}{70} = \boxed{\frac{1}{35}}$$



Conclude : $M = \frac{1}{12}$, $M_y = \frac{1}{20}$, $M_x = \frac{1}{35}$ (10)

Thus -

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{20}}{\frac{1}{12}} = \frac{12}{20} = \frac{3}{5}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{35}}{\frac{1}{12}} = \frac{12}{35}$$

Center of mass $(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35} \right)$

Ex: Same R and $\delta = 1$

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Find the KE of rotation of R about the y-axis, assuming a rotation rate of ω $\frac{\text{radians}}{\text{sec}}$

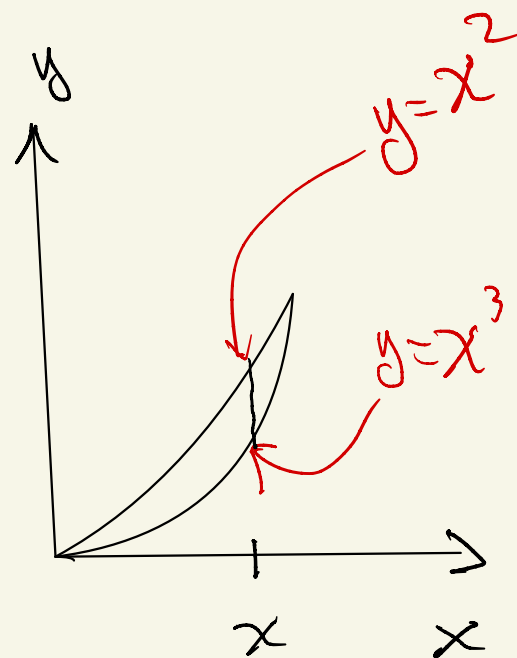
Soln

$$KE = \frac{1}{2} I_y \omega^2 = \frac{1}{60} \omega^2$$

$$I_y = \iint_R x^2 \delta(x, y) dA$$

$$= \int_0^1 \int_{x^3}^{x^2} x^2 dy dx = \int_0^1 x^2 y \Big|_{y=x^3}^{y=x^2} dx$$

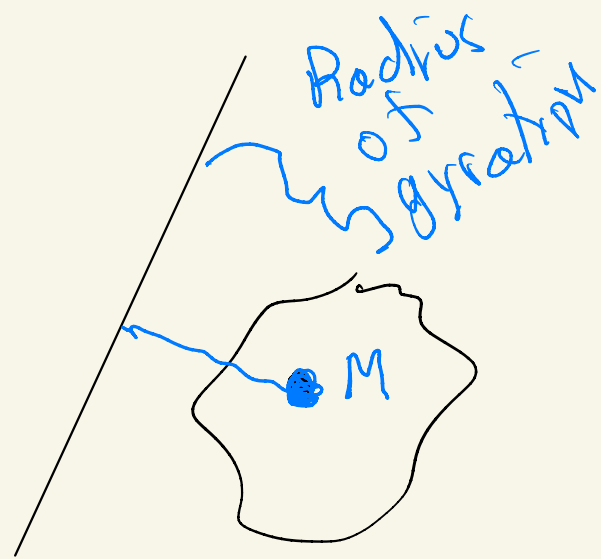
$$= \int_0^1 x^2 (x^2 - x^3) dx = \left[\frac{x^5}{5} - \frac{x^6}{6} \right]_0^1 = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$$



Ex. Find the radius of gyration of R about the y-axis —

Soln. The radius of gyration is the distance from the axis at which you get the same KE of rotation w/ all mass at that point —

Let r = radius of gyration. Then



$$\frac{1}{2} M (\underbrace{r\omega}_{\text{speed of } M})^2 = \frac{1}{2} I_L \omega^2$$

$$r^2 = \frac{I_L}{M} \Rightarrow$$

$$r = \sqrt{\frac{I_L}{M}}$$

For Us : $L \equiv y\text{-axis}$

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$$I_y = \frac{1}{30} \quad M = \frac{1}{12}$$

$$r = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{1/30}{1/12}} = \sqrt{\frac{12}{30}}$$

$$r = \sqrt{\frac{6}{15}}$$

Radius of
Gyration